Problem Set 1 – Statistical Physics B

Problem 1: Exact solution of the one-dimensional Ising model

In one spatial dimension the canonical partition function of the Ising model is

$$Z = \sum_{\{s_i\}} \exp\left[\sum_{i=1}^{N} (h s_i + K s_i s_{i+1})\right],$$

with $h = \beta\mu H$ and $K = \beta J$. Here H is an external magnetic field and J is a coupling constant between neighbouring spins. Consider periodic boundary conditions, $s_{N+1} = s_1$. This case can be solved analytically with the so-called transfer matrix method. This exercise will guide you through this procedure.

(a) Show that $\sum_{i=1}^{N} s_i = \sum_{i=1}^{N} (s_i + s_{i+1})/2$. Use this result to show that $Z = \mathrm{Tr}\,\mathbf{z}^N$, with the matrix $\mathbf{z}$ given by

$$\mathbf{z} = \begin{pmatrix} e^{-h+K} & e^{-K} \\ e^{-K} & e^{h+K} \end{pmatrix}.$$

(b) Show using the eigenrepresentation of $\mathbf{z}$ that $Z = \lambda_+^N + \lambda_-^N$, with $\lambda_\pm$ the eigenvalues of $\mathbf{z}$. We define $\lambda_+ > \lambda_-$.

(c) Determine $\lambda_\pm$ and show that the Helmholtz free energy for $N \to \infty$ becomes

$$\beta F/N = -K - \ln\left(\cosh h + \sqrt{\sinh^2 h + e^{-4K}}\right).$$

(d) Compute the average magnetization and show that it vanishes for $h \to 0^+$. Does the one-dimensional Ising model exhibit a phase transition?